TD 7: Partial-Order Reduction

Reminder:

(C0) $red(s) = \emptyset$ iff $en(s) = \emptyset$.

(C1) For every path $s \xrightarrow{a_1} s_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} s_n \xrightarrow{a} t$ in $\mathcal{K}$ (for any $n \geq 0$), if $a \notin red(s)$ and a depends on some action in $red(s)$ (i.e. there exists $b \in red(s)$ such that $(a, b) \notin I$), then there exists $1 \leq i \leq n$ such that $a_i \in red(s)$.

(C2) If $red(s) \neq en(s)$, then all actions in $red(s)$ are invisible.

(C3) For all cycles in the reduced system $\mathcal{K}'$, the following holds: if $a \in en(s)$ for some state s in the cycle, then $a \in red(s')$ for some (possibly other) state s' in the cycle.

Exercise 1. Consider the condition (C_1'): for any s with $red(s) \neq en(s)$, any a in $red(s)$ is independent from every b in $en(s) \setminus red(s)$.

1. Show that (C_1) implies (C_1').
2. Show that $(C_0), (C_1'), (C_2), (C_3)$ are not sufficient to ensure stuttering equivalence, i.e., that there exists a Kripke structure $\mathcal{K}$ and an assignment red satisfying conditions (C_0), (C_1'), (C_2), (C_3) but such that the reduced system $\mathcal{K}'$ induced by red is not stuttering equivalent to $\mathcal{K}$.

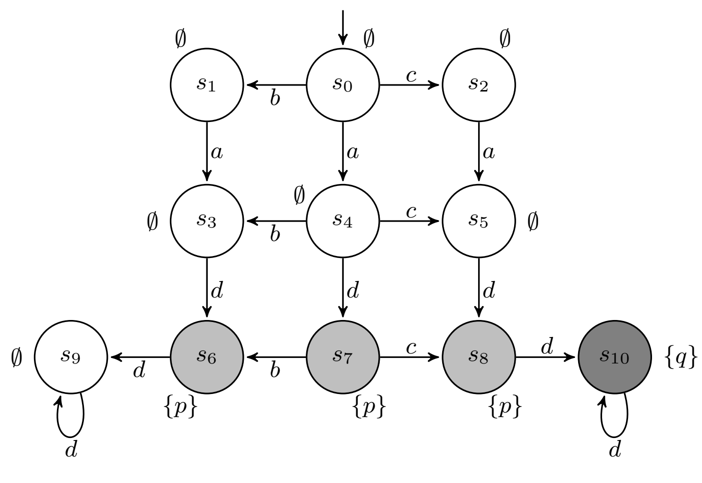
Exercise 2. Show that (C_0)–(C_2) is not sufficient to ensure stuttering equivalence.

Exercise 3. Show that checking condition (C1) is as hard as reachability checking.

More precisely, given an instance $\langle \mathcal{K}_1, a \rangle$ where $a \in AP$, show how to obtain an instance $\langle \mathcal{K}_2, red \rangle$ of the checking-condition-(C1) problem, such that $\mathcal{O}(|\mathcal{K}_2|) = |\mathcal{K}_1|$ and $\mathcal{K}_1 \models_\exists \mathsf{F}\, a$ *iff* the choice of the ample sets red in $\mathcal{K}_2$ violates condition (C1).

Hint: Start by adding a self-loop with a new action β to every state of $\mathcal{K}_1$. Also, letting s_0 be the initial state in $\mathcal{K}_1$, choose red such that $red(s_0) = \{\beta\}$.

Exercise 4. Consider the following system with $A = \{a, b, c, d\}$:



1. Let $red(s_0) = \{b, c\}$ and $red(s) = en(s)$ for $s \neq s_0$; show that this ample set assignment is compatible with C_0–C_3.

2. Exhibit a CTL(U) formula that distinguishes between the original system and its reduction.

3. Can you propose an assignment that also complies with C_4: if $red(s) \neq en(s)$, then $|red(s)| = 1$? You are not allowed to choose $red(s_0)$ to be $en(s_0)$.

Exercise 5. Let φ be an LTL formula. We define the X-depth $d_{\mathsf{X}}(\varphi)$ and the U-depth $d_{\mathsf{U}}(\varphi)$ of φ as the maximal nesting of X- or U-operators in φ:

$$
\begin{aligned}
d_{\mathsf{X}}(p) &= 0 & d_{\mathsf{U}}(p) &= 0\\
d_{\mathsf{X}}(\neg\varphi) &= d_{\mathsf{X}}(\varphi) & d_{\mathsf{U}}(\neg\varphi) &= d_{\mathsf{U}}(\varphi)\\
d_{\mathsf{X}}(\varphi \wedge \psi) &= \max(d_{\mathsf{X}}(\varphi), d_{\mathsf{X}}(\psi)) & d_{\mathsf{U}}(\varphi \wedge \psi) &= \max(d_{\mathsf{U}}(\varphi), d_{\mathsf{U}}(\psi))\\
d_{\mathsf{X}}(\mathsf{X}\,\varphi) &= 1 + d_{\mathsf{X}}(\varphi) & d_{\mathsf{U}}(\mathsf{X}\,\varphi) &= d_{\mathsf{U}}(\varphi)\\
d_{\mathsf{X}}(\varphi \,\mathsf{U}\, \psi) &= \max(d_{\mathsf{X}}(\varphi), d_{\mathsf{X}}(\psi)) & d_{\mathsf{U}}(\varphi \,\mathsf{U}\, \psi) &= 1 + \max(d_{\mathsf{U}}(\varphi), d_{\mathsf{U}}(\psi))
\end{aligned}
$$

We denote by $\mathrm{LTL}(\mathsf{U}^m, \mathsf{X}^n)$ the set of LTL formulas φ with $d_{\mathsf{X}}(\varphi) \leq n$ and $d_{\mathsf{U}}(\varphi) \leq m$, where $n = \infty$ or $m = \infty$ indicates no restriction of the operator in question.

1. We say that two words $w, w' \in \Sigma^\omega$ are *n-stutter-equivalent* if there exists letters $a_0, a_1, \ldots \in \Sigma$ and $f, g : \mathbb{N} \to \mathbb{N}^*$ such that $w = a_0^{f(0)} a_1^{f(1)} \ldots$, $w' = a_0^{g(0)} a_1^{g(1)} \ldots$, and for all $i \geq 0$, $a_i = a_{i+1}$ implies $a_i = a_j$ for all $j > i$, and $f(i) < n+1$ or $g(i) < n+1$ implies $f(i) = g(i)$.

   Show that for all $n \geq 0$ and $\varphi \in \mathrm{LTL}(\mathsf{U}^\infty, \mathsf{X}^n)$, $L(\varphi)$ is closed under n-stutter-equivalence.

2. A similar principle can be formulated when the U-depth is restricted, by considering stuttering of factors instead of letters. Show that for all $m \geq 1$ and $\varphi \in \mathrm{LTL}(\mathsf{U}^m, \mathsf{X}^0)$, for all $u, v \in \Sigma^*$ and $w \in \Sigma^\omega$, we have $uv^m w \in L(\varphi)$ iff $uv^{m+1} w \in L(\varphi)$.

3. Using the results above, show that the language $(aa|ab)^\omega$ cannot be defined by any LTL formula. (Remark: The language can, however, be accepted by a Büchi automaton.)